THE ETALE COHOMOLOGY OF THE GENERAL LINEAR GROUP OVER A FINITE FIELD AND THE DELIGNE AND LUSZTIG VARIETY

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ABSTRACT. Let $p \neq \ell$ be primes. We study the etale cohomology $H^*_{et}(BGL_n(\mathbb{F}_{p^s});\mathbb{Z}/\ell)$ over the algebraically closed field $\bar{\mathbb{F}}_p$ by using the stratification methods from Molina-Vistoli. To compute this cohomology, we use the Delinge-Lusztig variety.

1. Introduction

Let p and ℓ be primes with $p \neq \ell$. Let $G_n = GL_n(\mathbb{F}_q)$ the general linear group over a finite field $\mathbb{F}_q$ with $q = p^s$. Then Quillen computed the cohomology of this group in the famous paper [Qu].

Theorem 1.1. *(Quillen [Qu]) Let r be the smallest number such that $q^r - 1 = 0$ $mod(\ell)$. Then we have an isomorphism*

$$H^*(BG_n;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_r, ..., c_{r[n/r]}] \otimes \Delta(e_r, ..., e_{r[n/r]}) \quad (1.1)$$

where $|c_{rj}| = 2rj$, $|e_{rj}| = 2rj - 1$.

To prove this theorem, Quillen used the topological arguments, for example, the Eilenberg-Moore spectral sequences, and spaces of the kernel of the map $\psi^q - 1$ defined by the Adams operation. In this paper, we give an elementary algebraic proof for this theorem, in the sense without using the above topological arguments.

By induction on n and the equivariant cohomology theory (stratified methods) from Molina and Vistoli [Mo-Vi], [Vi], we can compute the etale cohomology over $k = \bar{\mathbb{F}}_p$, i.e., $H^*_{et}(BG_n;\mathbb{Z}/\ell) \cong (1.1)$. Then the base change theorem implies the Quillen theorem.

The Molina and Vistoli stratified methods also work for the motivic cohomology. Let $H^{*,*'}(-;\mathbb{Z}/\ell)$ be the motivic cohomology over the field $\bar{\mathbb{F}}_p$ and $0 \neq \tau \in H^{0,1}(Spec(\bar{\mathbb{F}}_p);\mathbb{Z}/\ell)$.

2000 *Mathematics Subject Classification.* Primary 11E72, 12G05; Secondary 55R35.

Key words and phrases. Deligne-Lusztig variety, classifying spaces, motivic cohomology.

Theorem 1.2. *We have an isomorphism* $H^{*,*'}(G_n;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\tau] \otimes$ (1.1) *with degree* $deg(c_{rj}) = (2rj, rj)$ *and* $deg(e_{rj}) = (2rj-1, rj)$.

To compute the equivariant cohomology, we consider the G_n-variety

$$Q' = Spec(k[x_1,...,x_n]/((-1)^{n-1}det(x_i^{q^{j-1}})^{q-1} = 1)),$$

and prove $Q'/G_n \cong \mathbb{A}^{n-1}$. This implies the equivariant cohomology

$$H^*_{G_n}(Q' \times_{\mu_{q^n-1}} \mathbb{G}_m; \mathbb{Z}/p) \cong \Delta(f), \quad |f| = 1.$$

The computation of the above isomorphism is the crucial point to compute $H^*_{G_n}(pt.;\mathbb{Z}/\ell) \cong H^*(BG_n;\mathbb{Z}/\ell)$.

Let G be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$, $q = p^r$, let $F\colon G \to G$ be the Frobenius and let G^F be the (finite) group of fixed points of F in G, e.g., $GL_n^F = G_n$ in our notation. In the paper [De-Lu], Deligne and Lusztig studied the representation theory of G^F over fields of characteristic 0. The main idea is to construct such representations in the ℓ-adic cohomology spaces ${H_c}^*(\tilde{X}(\dot{w}), \mathbb{Q}_\ell)$ of certain algebraic varieties $\tilde{X}(\dot{w})$ over $\mathbb{F}_q$, on which G^F acts. (see §6 for the definition of $\tilde{X}(\dot{w})$.)

For the $G = GL_n$ and$w = (1,\cdots,n)$, we see that $Q' \cong \tilde{X}(\dot{w})$. One of our theorems is to show $\tilde{X}(\dot{w})/G_n \cong \mathbb{A}^{n-1}$ for the above case by completely different arguments. The authors thank to Masaharu Kaneda and Shuichi Tsukuda for their useful suggestions.

2. Dickson Invariants

At first, we recall the Dickson algebra. Let us write $G_n = GL_n(\mathbb{F}_q)$. The Dickson algebra is the invariant ring of a polynomial of n variables under the usual G_n-action, namely,

$$\mathbb{F}_q[x_1,..,x_n]^{G_n} = \mathbb{F}_q[c_{n,0}, c_{n,1}, ..., c_{n,n-1}]$$

where each $c_{n,i}$ is defined by

$$\sum c_{n,i}X^{q^i} = \prod_{x\in\mathbb{F}_q\{x_1,...,x_n\}} (X+x) = \prod_{(\lambda_1,...,\lambda_n)\in\mathbb{F}_q^{\times n}} (X + \lambda_1 x_1 + ...\lambda_n x_n)$$

Hence the degree $|c_{n,i}| = q^n - q^i$ letting $|x_i| = 1$. Let us write $e_n = c_{n,0}^{1/(q-1)}$, namely,

$$e_n = (\prod_{0\neq x\in\mathbb{F}_q\{x_1,...,x_n\}} (x))^{1/(q-1)} = \begin{vmatrix} x_1 & x_1^q & \dots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \dots & x_2^{q^{n-1}} \\ \dots & \dots & \dots & \dots \\ x_n & x_n^q & \dots & x_n^{q^{n-1}} \end{vmatrix}.$$

Then each $c_{n,i}$ is written as

$$c_{n,s} = \begin{vmatrix} x_1 & \dots & \hat{x}_1^{q^s} & \dots & x_1^{q^n} \\ x_2 & \dots & \hat{x}_2^{q^s} & \dots & x_2^{q^n} \\ \dots & \dots & \dots & \dots & \dots \\ x_n & \dots & \hat{x}_n^{q^s} & \dots & x_n^{q^n} \end{vmatrix} /e(x).$$

Note that the Dickson algebra for $SG_n = SL_n(\mathbb{F}_q)$ is given as

$$\mathbb{F}_q[x_1, .., x_n]^{SG_n} = \mathbb{F}_q[e_n, c_{n,1}, ..., c_{n,n-1}].$$

Let us write $k = \bar{\mathbb{F}}_p$. We consider the algebraic variety

$$F = Spec(k[x_1, ..., x_n]/(e_n)).$$

We want to study the G_n-space structure of $X = X(n) = \mathbb{A}^n - \{0\}$ and $X(1) = X - F$. For this, we consider the following variety (the Deligne-Lusztig variety for $w = (1, ..., n)$, see §6 for details)

$$Q = Spec(k[x_1, ..., x_n]/(e_n - 1)).$$

Example. When $q = p$ and $n = 2$, we see

$$Q = \{(x, y)|x^p y - xy^p = 1\} \subset \mathbb{A}^2,$$

$$F = \{(x, y)|x^p y - xy^p = 0\} = \cup_{i \in \mathbb{F}_p \cup \{\infty\}} F_i$$

where $F_i = \{(x, ix)|x \in k\}$ and $F_\infty = \{(0, x)|x \in k\}$.

The corresponding projective variety $\bar{Q}$ is written

$$\bar{Q} = Proj(k[x_0, ..., x_n]/(e_n = x_0^{1+q+..+q^{n-1}})).$$

Lemma 2.1. *Let us write $q(n) = 1 + q + ... + q^{n-1} = (q^n - 1)/(q - 1)$. Then we have an isomorphism $Q \times_{\mu_{q(n)}} \mathbb{G}_m \cong X(1)$ of varieties.*

Proof. We consider the map

$$p : Q \times \mathbb{G}_m \to X(1) \quad by\ (x, t) \mapsto tx.$$

We see

$$e_n(p(x, t)) = e_n(tx_1, ..., tx_n) = t^{1+q+...+q^{n-1}} e_n(x_1, ..., x_n).$$

It is easily seen that this map is onto. Moreover if $x \in Q$ and $t \in \mu_{q(n)}$, then $p(x, t) = tx \in Q$. In fact $\mu_{q(n)}$ acts on Q. Since $p(x, t) = p(tx, 1)$, we have the isomorphism in this lemma. □

Remark 2.1 .It is immediate that the left SG_n-action and the right $\mu_{q(n)}$-action on Q is compatitive.i.e$(gx)\mu = g(x\mu) forg \in SG_n and \mu \in \mu_{q(n)}$.

Lemma 2.2. *We have $Q(\mathbb{F}_q) = \emptyset$.*

Proof. Let $(x_1, ..., x_n)$ be a $\mathbb{F}_q$-rational ponits. Then $x_i^q = x_i$. Hence we see

$$e_n = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1^q & x_2^q & \dots & x_n^q \\ \dots & \dots & \dots & \dots \\ x_1^{q^{n-1}} & x_2^{q^{n-1}} & \dots & x_n^{q^{n-1}} \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \end{vmatrix} = 0.$$

□

Lemma 2.3. *The group SG_n acts on Q freely.*

Proof. Assume that there is $0 \neq g \in G_n$ such that

$$gx = x \qquad for\ x \in Q \subset \mathbb{A}^n.$$

Then we can identify that x is an eigen vector for the (linear) action g with the eigen value 1. Hence we can take $x = (1, 0..., 0)$ after some change of basis. Of course $e_n(1, 0..., 0) = 0$ so $x \notin Q$. This is a contradiction. □

The group SG_n acts freely on the (smooth) variety Q. Hence Q/SG_n exists as a variety and we have

$$Q/SG_n = Spec(A^{SG_n}) \quad for\ A = k[x_1, ..., x_n]/(e_n - 1).$$

Theorem 2.4. *We have an isomorphism*

$$A^{SG_n} \cong k[c_{n,1}, ..., c_{n,n-1}] \quad i.e., \quad Q/SG_n \cong \mathbb{A}^{n-1}.$$

Proof. It is almost immediate

$$k[c_{n,1}, ..., c_{n,n-1}] \subset A^{SG_n}.$$

The coordinate ring $\bar{A}$ of the Zariski closure $\bar{Q}$ of Q in $\mathbb{P}^n$ is given as

$$\bar{A} = k[x_0, ..., x_n]/(e_n = x_0^{1+q+..+q^{n-1}}).$$

Of course, the coordinate ring $\bar{B}$ of the closure of $Spec(k[c_{n,1}, ..., c_{n,n-1}])$ in $\bar{Q}$ is given as

$$\bar{B} = k[x_0, c_{n,1}, ..., c_{n,n-1}].$$

Next we compute the Poincare polynomials of $\bar{A}$ and $\bar{A}^{SG_n}$;

$$PS(\bar{A}) = (1-t^{1+q+...+q^{n-1}})/(1-t)^{n+1} = (1+t+...+t^{q+...+q^{n-1}})/(1-t)^n,$$

$$PS(\bar{B}) = 1/(1-t)(1-t^{|c_{n,1}|})...(1-t^{|c_{n,n-1}|})$$
$$= (1+t+...+t^{|c_{n,1}|-1})^{-1}...(1+t+...+t^{|c_{n,n-1}|-1})^{-1}/(1-t)^n.$$

Hence we get

$$PS(\bar{A})/PS(\bar{B}) = (1+t+...t^{|c_{n,1}|-1})...(1+t+...t^{|c_{n,n-1}|-1})$$
$$\times(1+t+...+t^{q+..+q^{n-1}}).$$

Thus we know

$$rank(PS(\bar{A})/PS(\bar{B})) = |c_{n,1}| \times ... \times |c_{n,n-1}| \times (1 + q + ... + q^{n-1})$$
$$= (q^n - q^1)...(q^n - q^{n-1})((q^n - 1)/(q-1)) = |SG_n|.$$

On the other hand $c_{n,1}, ..., c_{n,n-1}$ is regular sequence in $\bar{A}$. Hence $\bar{A}$ is $\bar{B}$-free, that is

$$\bar{A} = \bar{B}\{x_1, ..., x_m\}$$

where $m = |SG_n|$ from the results using the Poincare polynomials above.

Let $\pi : Q \to Q/SG_n$ be the projection. Since π is etale, for all $x \in Q$, the local ring O_x is $O_{\pi(x)}$-free, and $rank_{O_{x'}}(O_x) = |SG_n|$. Thus we get the desired result $A^{SG_n} = k[c_{n,1}, ..., c_{n,n-1}]$. □

Similarly, we can prove

Theorem 2.5. *Let* $A' = k[x_1, ..., x_n]/(e_n^{q-1} - 1)$ *and* $Q' = Spec(A')$. *Then we have an isomorphism*

$$(A')^{G_n} \cong k[c_{n,1}, ..., c_{n,n-1}] \quad i.e., \quad Q'/G_n \cong \mathbb{A}^{n-1}.$$

In §7 below, we give a complete different proof of the above theorem.

3. Equivariant cohomology

For a smooth algebraic variety X over $k = \bar{\mathbb{F}}_p$, we consider the *mod* ℓ etale cohomology for $\ell \neq p$. Let us write simply

$$H^*(X) = H^*_{et}(X; \mathbb{Z}/\ell).$$

Let $\rho : G \to W = \mathbb{A}^n$ a faithful representation. Let $V_n = W - S$ be an open set of W such that G act freely V_n where $codim_W S > n \geq 2$. Then the classifying space BG of G is defined as $colim_{n\to\infty}(V_n/G)$. Let X be a smooth G-variety. Then we can define the equivariant cohomology ([Vi], [Mo-Vi])

$$H^*_G(X) = lim_n H^*_{et}(V_n \times_G X; \mathbb{Z}/\ell).$$

Of course $H^*_G(pt.) = H^*(BG) = H^*_{et}(BG; \mathbb{Z}/\ell)$.

One of the most useful facts in equivariant cohomology theories is the following localized exact sequence. Let $i : Y \subset X$ be a regular closed inclusion of G-varieties, of $codim_X(Y) = c$ and $j : U = X - Y \subset X$. Then there is a long exact sequence

$$\to H^{*-2c}_G(Y) \xrightarrow{i_*} H^*_G(X) \xrightarrow{j^*} H^*_G(U) \xrightarrow{\delta} H^{*-2c+1}_G(Y) \to ...$$

Now we apply the above exact sequence for concrete cases. We consider the case $G = G_n = GL_n(\mathbb{F}_q)$. Recall

$$F = Spec(k[x_1, ..., x_n]/(e_n^{q-1})) = \cup_{\lambda=(\lambda_1,...,\lambda_n)\neq 0}(F_\lambda)$$

where $F_\lambda = \{(x_1,...,x_n)|\lambda_1 x_1 + ... + \lambda_n x_n = 0\} \subset \mathbb{A}^n$.

Let $F(1) = F$ and $F(2)$ be the ($codim = 1$) set of singular points in $F(1)$, namely, $F(2) = \cup F_{\lambda,\mu}$ with

$$F_{\lambda,\mu} = \begin{cases} F_\lambda \cap F_\mu \quad if\ F_\lambda \neq F_\mu \\ \emptyset \quad if\ F_\lambda = F_\mu. \end{cases}$$

Similarly, we define $F(i)$ as the variety defined by the set of $codim_{\mathbb{A}^n} F(i) = i$. Let us write $X(i) = X - F(i)$. Thus we have a sequence of the algebraic sets

$$F(1) \supset F(2) \supset ... \supset F(n) = \{0\} \supset F(n+1) = \emptyset,$$

$$X - F(1) = X(1) \subset X(2) \subset ... \subset X(n) = \mathbb{A}^n - \{0\} \subset X(n+1) = \mathbb{A}^n.$$

Therefore we have the long exact sequences

$$\to H^{*-2}_{G_n}(F(1) - F(2)) \xrightarrow{i_*} H^*_{G_n}(X(2)) \xrightarrow{j^*} H^*_{G_n}(X(1)) \xrightarrow{\delta} ...,$$

....

$$\to H^{*-2i}_{G_n}(F(i) - F(i+1)) \xrightarrow{i_*} H^*_{G_n}(X(i+1)) \xrightarrow{j^*} H^*_{G_n}(X(i)) \xrightarrow{\delta} ...,$$

....

$$\to H^{*-2n}_{G_n}(F(n) - F(n+1)) \xrightarrow{i_*} H^*_{G_n}(X(n+1)) \xrightarrow{j^*} H^*_{G_n}(X(n)) \xrightarrow{\delta} ...$$

Lemma 3.1. *We have $H^*_{G_n}(X(1)) \cong \Lambda(f)$ with $|f| = 1$.*

Proof. ¿From the G_n version (but not SG_n) of Lemma 2.1, we have

$$X(1) \cong Q' \times_{\mu_{q^n-1}} \mathbb{G}_m.$$

Hence we can compute the equivariant cohomology from Theorem 2.5,Lemma 2.3 and Remark 2.1

$$H^*_{G_n}(X(1)) \cong H^*(X(1)/G_n)$$

$$\cong H^*(Q'/G_n \times_{\mu_{q^n-1}} \mathbb{G}_m) \cong H^*(\mathbb{A}^{n-1} \times_{\mu_{q^n-1}} \mathbb{G}_m)$$

$$\cong H_{\mu_{q^n-1}}(\mathbb{G}_m) \cong \Lambda(f) \quad |f| = 1.$$

□

Lemma 3.2. *We have an isomorphism*

$$H^*_{G_n}(F(i) - F(i+1)) \cong H^*(BG_i) \otimes \Lambda(f)$$

Proof. Each irreducible component of $F(i)$ is a $codim = i$ subspace, which is also identified an element of the Grassmannian. Hence we can write

$$F(i) - F(i+1) \cong \amalg_{\bar{g}\in G_n/(P_{i,n-i})} g(\mathbb{A}^{n-i} - F(1)')$$

where $g \in G_n$ is a representative element of $\bar{g}$, $F(1)' = Spec(k[x_1, ..., x_{n-i}]/(e_{n-i}^{q-1})$ and $P_{i,n-i}$ is the parabolic subgroup

$$P_{i,n-i} = (G_i \times G_{n-i}) \ltimes U_{i,n-i}(\mathbb{F}_q) \cong \{ \begin{pmatrix} G_i & * \\ 0 & G_{n-i} \end{pmatrix} | * \in U_{i,n-i}(\mathbb{F}_q)\}.$$

Since the stabilizer group of $X(1)' = \mathbb{A}^{n-i} - F(1)'$is $P_{i,n-i}$, we note from [Vi] that $H^*_{G_n}(F(i) - F(i+1)) \cong H^*_{P_{i,n-1}}(X(1)') \cong H^*_{G_i\times G_{n-i}}(X(1)')$.

Hence we can compute (for $* < N$)

$$H^*_{G_n}(F(i) - F(i+1)) \cong H^*(V'_N \times V''_N \times_{G_i\times G_{n-i}} X(1)').$$

$$\cong H^*((V'_N/G_i) \times V''_N \times_{G_{n-i}} X(1)').$$

$$\cong H^*_{G_i} \otimes H^*_{G_{n-i}}(X(1)').$$

Here $X(1)'$ is the $(n-i)$-dimensional version of $X(1)$, and we identify $V_N \cong V'_N \times V''_N$ where G_i acts freely on V'_N and so on. ¿From the preceding lemma, we know $H^*_{G_{n-i}}(X(1)') \cong \Lambda(f)$. □

Let r be the smallest number such that $q^r - 1 = 0\ mod(\ell)$. Recall that

$$|G_n| = (q^n - 1)(q^n - q)...(q^n - q^{n-1}).$$

Hence if $n < r$, then $H^*(BG_n) \cong \mathbb{Z}/\ell$, and hence $H^*_{G_n}(F(i) - F(i+1)) \cong \Lambda(f)$ for $i \le n$.

The cohomology of BGL_n is the same as that of $BGL_n(\mathbb{C})$, i.e.,

$$H^*(BGL_n) \cong \mathbb{Z}/\ell[c_1, ..., c_n].$$

The Frobenius map F acts on this cohomology by $c_i \mapsto q^i c_i$. Recall that the Lang map induces a principal G_n-bundle

$$G_n \to GL_n \xrightarrow{L} GL_n$$

where $L(g) = g^{-1}F(g)$. Thus we have a map

$$H^*(BGL_n)/((q^i - 1)c_i) \cong \mathbb{Z}/\ell[c_r, ..., c_{[n/r]r}] \to H^*(BG_n).$$

Lemma 3.3. *If $r = 1$, then we have an isomorphism*

$$H^*(BG_n) \cong \mathbb{Z}/\ell[c_1, ..., c_n] \otimes \Delta(e_1, ..., e_n).$$

Proof. We prove by induction on n. Assume that

$$H^*(BG_i) \cong \mathbb{Z}/\ell[c_1,...,c_i] \otimes \Delta(e_1,...,e_i) \quad for\ i < n.$$

We consider the long exact sequence

$$\to H^{*-2i}_{G_n}(F(i)-F(i+1)) \xrightarrow{i_*} H^*_{G_n}(X(i+1)) \xrightarrow{j^*} H^*_{G_n}(X(i)) \xrightarrow{\delta} ...$$

Here we use induction on i, and assume that

$$H^*_{G_n}(X(i)) \cong H^*_{G_{i-1}} \otimes \Lambda(e_i)$$
$$\cong \mathbb{Z}/\ell[c_1,...,c_{i-1}] \otimes \Delta(e_1,...,e_{i-1}) \otimes \Lambda(e_i).$$

(Letting $e_1 = f$, we have the case $i = 1$ from Lemma 3.1.) ¿From the preceding lemma, we still see

$$H^*_{G_n}(F(i)-F(i+1)) \cong H^*_{G_i} \otimes \Lambda(f)$$
$$\cong \mathbb{Z}/\ell[c_1,...,c_i] \otimes \Delta(e_1,...,e_i) \otimes \Lambda(f).$$

In the above long exact sequence,the map j^* is an epimorphism for $* < 2i-1$, because $H^{minus}(F(i)-F(i+1)) = 0$. But $H^*_{G_n}(X(i))$ is multiplicatively generated by the elements of $dim \leq 2i-2$ and e_i. By dimensional reason, we see

$$\delta(e_i) = 1 \quad or \quad \delta(e_i) = 0.$$

Of course if $\delta(e_i) = 0$, then $\delta = 0$ for all $* \geq 0$.

Consider the restriction map $H^*_{G_n}(X(i+1)) \to H^*_{G_i}(\mathbb{A}^i)$ which is induced from $X(i+1) = \mathbb{A}^n - F(i+1) \supset \mathbb{A}^i$. Since $|G_i| = (q^{ir}-1)q|G_{i-1}|$, the ℓ-Sylow subgroup of G_i is different from that of G_{i-1}, (More precisely, $rank_\ell G_i > rank_\ell G_{i-1}$.) So from the Quillen theorem, the Krull dimension of $H_{G_n}(X(i+1))$ is larger than that of $H^*_{G_n}(X(i))$. This fact implies $i_*(1) = c_i$. (Let $p : V \to X$ be a j-dimensional bundle and $i : X \to V$ a section. Then the Chern class c_j is defined as $i^*i_*(1)$.) Thus we see $\delta(e_i) = 0$.

Therefore we have the short exact sequence

$$0 \to H^*_{G_i} \otimes \Lambda(f) \xrightarrow{i_*} H^*_{G_n}(X(i+1)) \xrightarrow{j^*} H^*_{G_{i-1}} \otimes \Lambda(e_i) \to 0,$$

namely, we have an isomorphism

$$grH^*_{G_n}(X(i+1)) \cong \mathbb{Z}/\ell[c_1,...,c_{i-1}] \otimes \Delta(e_1,...,e_i)$$
$$\otimes(\mathbb{Z}/\ell[c_i]\{i_*(1) = c_i, i_*(f)\} \oplus \mathbb{Z}/\ell\{1\}).$$

Let us write $i_*(f) = e_{i+1}$. Then $H^*_{G_n}(X(i+1))$ is the desired form

$$H^*_{G_n}(X(i+1)) \cong \mathbb{Z}/\ell[c_1,...,c_{i-1}] \otimes \Delta(e_1,...,e_i)$$
$$\otimes(\mathbb{Z}/\ell[c_i]\{c_i, e_{i+1}\} \oplus \mathbb{Z}/\ell\{1\})$$
$$\cong \mathbb{Z}/\ell[c_1,...,c_i] \otimes \Delta(e_1,...,e_i) \otimes \Lambda(e_{i+1}).$$

Thus we can see the desired result $H^*_{G_n}(X(n+1)) \cong H^*(BG_n)$. □

Remark. In the above proof, to see $i_*(1) = c_i$ we used the Krull dimesion (by Quillen). However there is more natural argument (see Proposion 4.2 in the next section) where the properties of the maximal torus $T(\dot{w})$ are used.

Theorem 3.4. *We have the isomorphism*

$$H^*(BG_n) \cong \mathbb{Z}/\ell[c_r, ..., c_{[n/r]r}] \otimes \Delta(e_r, ..., e_{[n/r]r}).$$

Proof. We prove the theorem by induction on n. Assume that

$$H^*(BG_i) \cong \mathbb{Z}/\ell[c_r, ..., c_{[n/i]r}] \otimes \Delta(e_r, ..., e_{[n/r]r}) \quad for\ i < n.$$

We also consider the long exact sequence

$$\to H^{*-2i}_{G_n}(F(i) - F(i+1)) \xrightarrow{i_*} H^*_{G_n}(X(i+1)) \xrightarrow{j^*} H^*_{G_n}(X(i)) \xrightarrow{\delta} ...$$

Here we use induction on i, and assume $H^*_{G_n}(X(i)) \cong H^*_{G_{i-1}} \otimes \Lambda(e_i)$.

¿From Lemma 3.2, we still see

$$H^*_{G_n}(F(i) - F(i+1)) \cong H^*_{G_i} \otimes \Lambda(f).$$

By dimensional reason, we see $\delta(e_i) = 1 \quad or \quad \delta(e_i) = 0$.

Now we consider the case $r \geq 2$ and $mr < i \leq mr + r - 1$. This case we still assume

$$H^*_{G_i} \cong H^*_{G_{i-1}} \cong H^*_{G_{mr}} \cong \mathbb{Z}/\ell[c_r, ..., c_{mr}] \otimes \Delta(e_r, ..., e_{mr}).$$

Hence the above exact sequence is written as

$$\to H^*_{G_{mr}} \otimes \Lambda(f) \xrightarrow{i_*} H^*_{G_n}(X(i+1)) \xrightarrow{j^*} H^*_{G_{mr}} \otimes \Lambda(e_i) \to ....$$

The ℓ- Sylow subgroup of G_i and G_{i-1} are the same, and hence $c_i = 0$ in $H^*_{G_i}$. (See also Proposition 4.2 below.) This means $\delta(e_i) = 1$ (Of course $\delta(1) = 0$).

Hence we have the isomorphism

$$H^*_{G_n}(X(i+1)) \cong H^*_{G_{mr}}\{1, i_*(f)\} \cong H^*_{G_{mr}}\{1, e_{i+1}\} \cong H^*_{G_i} \otimes \Lambda(e_{i+1}).$$

Other parts of the proof are almost the same as in the case $r = 1$. □

4. MAXIMAL TORUS AND SL_n

Let r be the smallest positive integer such that $q^r - 1 = 0\ (mod(\ell))$. Let $w = (1, 2, .., r) \in S_r$ and $G_r = GL_r(\mathbb{F}_q) = GL_r^F$ for the Frobenius map $F : x \mapsto x^q$. For a matrix $A = (a_{i,j}) \in GL_n$, the adjoint action is given as

$$ad(w)F(A) = wFw^{-1}(a_{i,j}) = (b_{i,j}) \quad with\ b_{i,j} = a^q_{i-1,j-1}.$$

Let $T(w)$ be the maximal torus $T^* \subset GL_r$, for which the Frobenius is given as $ad(w)F$ (see the next section for details) so that

$$T(w)^F = \{t \in T^* | ad(w)F(t) = t\}$$

$$\cong \{x \in \mathbb{F}_{q^r}^* | (x, x^q, ..., x^{q^{r-1}}) \in T^*\} \cong \mathbb{F}_{q^r}^*.$$

Take $H^*(BT^*) \cong \mathbb{Z}/\ell[t_1, .., t_r]$. Let $i : T(w)^F \subset T^*$. Then we can take the ring generator $t \in H^2(BT(w)^F)$ such that $i^* t_i = q^{i-1} t$.

Lemma 4.1. *The following map is injective*

$$H^*(BGL_r)/((q^i - 1)c_i) \cong \mathbb{Z}/\ell[c_r] \to H^*(BG_r).$$

Proof. It is enough to prove that for the map

$$i^* : H^*(BGL_r) \to H^*(BG_r) \to H^*(BT(w)^F) \cong H^*(\mathbb{F}_{q^r}^*),$$

we can see $i^* c_1 = ... = i^* c_{r-1} = 0$, and $i^* c_r = (-1)^r t^r$.

Let s_i be the i-th elementary symmetric function of variables $t_1, ..., t_r$, namely,

$$(X - t_1)(X - t_2)...(X - t_r) = X^r + s_1 X^{r-1} + ... + s_r.$$

Since $i^*(t_i) = q^{i-1} t$, we see that

$$(X - t)(X - qt)...(X - q^{r-1} t) = X^n + i^*(s_1) X^{r-1} + ... + i_*(s_r).$$

On the other hand, the polynomial $X^r - t^r$ has its roots $X = t, qt, ..., q^{r-1}t$. Hence we see that the above formula is $X^r - t^r$. It implies the assertion above. □

Proposition 4.2. *The following map is injective*

$$H^*(BGL_n)^F \cong \mathbb{Z}/\ell[c_r, ..., c_{[n/r]r}] \to H^*(BG_r).$$

Proof. Let $k = [n/r]$. let us take

$$w = (1, ..., r)(r + 1, ..., 2r)...((k - 1)r + 1, ..., kr).$$

We consider the map

$$i^* : H^*(BGL_n) \to H^*(BG_n) \to H^*(BT(w)^F) \cong H^*(B(\mathbb{F}_{q^r}^* \times ... \times \mathbb{F}_{q^r}^*)).$$

We chose $t_i \in H^2(BT)$ $(1 \le i \le n)$ and $t_j' \in H^2(BT(w)^F)$ $(1 \le j \le k)$ such as $i^* t_1 = t_1', i^* t_2 = q t_1', ...$ Then the arguments similar to the proof of the preceding lemma, we have

$$X^n + i^*(c_1) X^{r-1} + ... + i_*(c_r) = (X^r \pm (t_1')^r)...(X^r \pm (t_k')^r).$$

Thus we get the result. □

Now we consider the case $G = SL_n$. Write $SL_n(\mathbb{F}_q)$ by SG_n.

Lemma 4.3. *If $r \geq 2$, then, the following map is injective*

$$H^*(BSL_r)^F \cong \mathbb{Z}/\ell[c_r] \to H^*(BSG_r).$$

Proof. Let $w = (1, ..., r)$ and recall $q(r) = 1 + q + ... + q^{r-1}$. Then the maximal torus of SG_r is written

$$ST^*(w)^F \cong \{t \in F_{q^r}^* | (x, ..., x^{q^{r-1}}) \in T^*,\ x^{q(r)} = 1\} \cong \mathbb{Z}/q(r).$$

We consider the map as the case G_r

$$i^* : H^*(BSL_r) \to H^*(BSG_r) \to H^*(BST(w)^F)$$

$$\cong H^*(B\mathbb{Z}/q(r)) \cong \mathbb{Z}/\ell[t] \otimes \Lambda(v).$$

Let us write $H^*(BST^*) \cong \mathbb{Z}/\ell[t_1, ..., t_r]/(t_1 + .. + t_r)$. Then we also see that $i^*(t_i) = q^{i-1}t$ (note $\sum q^{i-1} = q(r) = 0 \in \mathbb{Z}/\ell$). The arguments in the proof of Lemma implies this lemma. □

Proposition 4.4. *For the case $r \geq 2$, the following map is injective*

$$H^*(BGL_n)^F \cong \mathbb{Z}/\ell[c_r, ..., c_{[n/r]r}] \to H^*(BSG_n).$$

When $r = 1$, the map $\mathbb{Z}/\ell[c_2, ..., c_n] \to H^*(BSG_n)$ is injective.

Proof. The maximal torus of SG_n is written

$$ST^*(w)^F \cong \{t \in F_{q^r}^* | (x_1, ..., x_1^{q^{r-1}}, ..., x_k, ..., x_k^{q^{r-1}})) \in T^*,\ (x_1 ... x_k)^{q(r)} = 1\}.$$

We can get the result as the case G_n. When $r = 1$, note that $c_1 = t_1 + ... + t_n = 0$ still in $H^*(BST^*)$. □

Theorem 4.5. *For the case $r \geq 2$, we have the isomorphism $H^*(BSG_n) \cong H^*(BG_n)$. When $r = 1$, we have*

$$H^*(BG_n; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_2, ..., c_n] \otimes \Delta(e_2, ..., e_n).$$

An outline of the proof. Almost arguments work as the case G_n. For example, in the proof of Lemma 3.2, for $G = G_n$, we showed

$$F(i) - F(i+1) \cong G_n/(P_{i,n-i}) \times (\mathbb{A}^{n-i} - F(1)')$$

where $P_{i,n-i}$ is the parabolic subgroup $(G_i \times G_{n-i}) \ltimes U_{i,n-i}$. We must consider the SG_n-version

$$SG_n/S(G_i \times G_{n-i}) \ltimes U_{i,n-i}(\mathbb{A}^{n-i} - F(1)').$$

Here we can reduce $S(G_i \times G_{n-i})$ to the case $G_i \ltimes SG_{n-i}$. Then the inductive arguments work also this case. □

5. MOTIVIC COHOMOLOGY

In this section, we consider the motivic version of preceding section. Let us write

$$H_G^{*,*'}(X) = H_G^{*,*'}(X;\mathbb{Z}/p)$$

the (equivariant) motivic cohomology over the field $k = \bar{\mathbb{F}}_p$. Then we have the long exact sequence

$$\to H_{G_n}^{*-2i,*'-i}(F(i)-F(i+1)) \xrightarrow{i_*} H_{G_n}^{*,*'}(X(i+1)) \xrightarrow{j^*} H_{G_n}^{*,*'}(X(i)) \xrightarrow{\delta} .$$

However we note the following fact: the projection

$$\begin{aligned} V_N'' \times_{G_{n-i}} (\mathbb{A}^{n-i} - F(1)') \to & \mathbb{A}^{n-i} - F(1)'/G_{n-i} \\ & \cong \mathbb{A}^{n-i-1} \times_{\mu_{q^{n-i}-1}} \mathbb{G}_m \to \mathbb{G}_m/\mu_{q^{n-i}-1} \cong \mathbb{G}_m \end{aligned}$$

is an $\mathbb{A}^1$-homotopy equivalence when we replace V_N'' as a suitable large G_{n-i}-vector space. Then Lemma 3.2 holds for the motivic cohomology. Then the most arguments in the preceding sections also work for the motivic cohomology with the degree

$$deg(c_i) = (2i, i), \qquad deg(e_i) = (2i-1, i).$$

Thus we get Theorem 1.2 in the introduction.

6. THE DELIGNE-LUSZTIG THEORY

Let G be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$, $q = p^r$, let $F\colon G \to G$ be the Frobenius map and let G^F be the (finite) group of fixed points of F in G.

In the paper [De-Lu], Deligne and Lusztig studied the representation theory of G^F over fields of characteristic 0. The main idea is to construct such representations in the ℓ-adic cohomology spaces of certain algebraic varieties $\tilde{X}(\dot{w})$ over $\mathbb{F}_q$, on which G^F acts.

Fix a Borel subgroup $B^* \subset G$ and a maximal $\mathbb{F}_q$-split torus $T^* \subset B^*$, both defined over $\mathbb{F}_q$. Let W be the Weyl group of T^* and

$$G = \bigcup_{w\in W} B^*\dot{w}B^* \qquad (disjoint\ union)$$

be the Bruhat decomposition, $\dot{w}$ being a representative of $w \in W$ in the normalizer of T^*. Let X be the variety of all Borel subgroups of G. This is a smooth scheme over $\mathbb{F}_q$, on which the Frobenius element F acts. Any $B \in X$ is of the form $B = gB^*g^{-1} = adgB^*$, where $g \in G$ is determined by B up to right multiplication by an element of B^*. Let $X(w) \subset X$ be the locally closed subscheme consisting of all Borel subgroups $B = gB^*g^{-1}$ such that $g^{-1}F(g) \in B^*\dot{w}B^*$, namely,

$$(6.1) \quad X(w) = \{g \in G | g^{-1}F(g) \in B^*\dot{w}B^*\}/B^*$$

$$\cong \{g \in G | g^{-1}F(g) \in \dot{w}B^*\}/(B^* \cap ad\dot{w}B^*).$$

(Borel groups $ad(g)B^*$ and $ad(g)FB^*$ are called in relative position w if $g \in X(\dot{w})$.)

For any $w \in W$, let $T(w)$ be the torus T^*, for which the Frobenius map is given by $ad(w)F$ so that

$$(6.2) \quad T(w)^F = \{t \in T^* | ad(w)F(t) = t\}.$$

Hence $T(w)^F$ is isomorphic to the set of $\mathbb{F}_q$-points of a torus $T(w) \subset G$, defined over $\mathbb{F}_q$.

Let U^* be the unipotent radical of B^*. For any $B \in X$ let $E(B) = \{g \in G | gB^*g^{-1} = B\}/U^*$. The Frobenius map induces a map $F\colon E(B) \to E(F(B))$. Let $E(B, \dot{w}) = \{u \in E(B) | F(u) = u\dot{w}\}$. For $B \in X(w)$ the sets $E(B, \dot{w})$ are the fibers of a map $\pi\colon \tilde{X}(\dot{w}) \to X(w)$, where $\tilde{X}(\dot{w})$ is a right principal homogeneous space of $T(w)^F$ over $X(w)$. The groups G^F and $T(w)^F$ act on $\tilde{X}(\dot{w})$ and these actions commute. Thus we have the isomorphism

$$(6.3) \quad \tilde{X}(\dot{w}) \cong \{g \in G | g^{-1}F(g) \in \dot{w}U^*\}/(U^* \cap ad\dot{w}U^*).$$

Now let ℓ be a prime distinct from p, and $\mathbb{Q}_\ell$ be the algebraic closure of the field of ℓ-adic numbers. Deligne-Lusztig consider the actions of G^F and $T(w)^F$ on the ℓ-adic cohomology ${H_c}^*(\tilde{X}(\dot{w}), \mathbb{Q}_\ell)$ with compact support. For any $\theta \in Hom(T(w)^F, \mathbb{Q}_\ell)$, let $H_c^*(\tilde{X}(\dot{w}), \mathbb{Q}_\ell)_\theta$ be the subspace of ${H_c}^*(\tilde{X}(\dot{w}), \mathbb{Q}_\ell)$ on which $T(w)^F$ acts by θ. This is a G^F-module.

The main subject of the paper [De-Lu] is the study of virtual representations $R^\theta(w) = \sum_i (-1)^i {H_c}^i(\tilde{X}(\dot{w}), \mathbb{Q}_\ell)_\theta$ (it can be shown that the right hand side is independent of the lifting $\dot{w}$ of w).

Example. (See 2.1 in [De-Lu].) Let V be an n-dimensional vector space over k and put $G = GL(V)$. We may take a basis such that a maximal torus $T \cong \mathbb{G}_m^n$ and the Weyl group $W \cong S_n$; the symmetric group of n-letters. Then $X = G/B$ is the space of complete flags

$$D \ : \ D_0 = 0 \subset D_1 \subset ... \subset D_{n-1} \subset D_n = V$$

with $dim D_i = i$. The space $E = G/T$ is the space of complete flags marked by nonzero vector $e_i \in D_i/D_{i-1}$, where T acts on E by $(D, (e_i))(t_i) = (D, (t_i e_i))$.

Let $w = (1, ..., n)$. Then two flags D' and D'' are relative position w (for details see 1.2 in [De-Lu]) if and only if

$$D''_i + D'_i = D'_{i+1} \ (1 \le i < n-1), \quad D''_{n-1} + D^i_1 = V.$$

Hence D and FD are in relative position w, if and only if

$$D_1 \subset D_1 + FD_1 \subset D_1 + FD_1 + F^2 D_1 \subset ...$$

and $V = \oplus^{n+1} F^i D_1$. A marking e of F is given such that $F(e) = e \cdot \dot{w}$ if and only if

$$e_2 = F(e_1)(mod(e_1)), \quad ..., \quad e_n = F^{n-1}(e_1)(mod(e_1, ..., F^{n-2}(e_1))$$
$$and \qquad e_1 = F^n(e_1)(mod(e_1, ..., F^{n-1}(e_1));$$

Hence the mark e is defined by $e_1 \in D_1$ with the condition that

$$F(e_1 \wedge F(e_1) \wedge ... \wedge F^{n-1}(e_1)) = (-1)^{n-1}(e_1 \wedge F(e_1) \wedge ... \wedge F^{n-1}(e_1)).$$

If (x_i) are the coordinate of e_1, the above condition can be rewritten

$$(6.4) \quad (-1)^{n-1}(det(x_i^{q^{j-1}})_{1\le i,j\le n})^{q-1} = 1.$$

Hence the map (D_1, e_1) induces an isomorphism of $\tilde{X}(\dot{w})$ with the affine hypersurface (6.4). Note that this hypersurface is stable under $x \mapsto tx$ for $t \in F_{q^n}^*$, and this is the action of $T(w)^F$.

Recall that $(det(x_i^{q^{j-1}})_{1\le i,j\le n})$ is written by e_n in §2. Thus we have

Theorem 6.1. *The variety Q' in Theorem 2.5 in §2 is isomorphic to $\tilde{X}(\dot{w})$.*

In the next section, we will give a complete different proof of the above theorem.

7. The Deligne-Lusztig variety $\tilde{X}(\dot{w}_n)$

In 1.11.4 in [De-Lu], Deligne and Lustig prove the following theorem

Theorem 7.1.

$$G_n \setminus \tilde{X}(\dot{w}_n) \cong U^*/(U^* \cap ad(\dot{w}_n)U^*).$$

We will give a complete different proof of the above theorem and Theorem 2.4 by using Dickson invaraints for $G = GL_n(\mathbb{F}_q)$ and $w_n = (1, ..., n)$.

Take an adequate basis of the n dimensional vector space such that

$$w_n = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \qquad U^* = \{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 \end{pmatrix} | * \in \bar{F}_p \}.$$

Let $x_{i,j}(a) = 1 + ae_{i,j}$ where $e_{i,j}$ is the elementary matrix with 1 in (i,j)-entry and 0 otherwise. Then U^* is generated by $x_{i,j}(a)$,

$$U^* = \langle x_{i,j}(a) | 1 \le i < j \le n \mid a \in \bar{F}_p \rangle$$

with the relation

$$x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b), \quad [x_{i,j}(a), x_{k,l}(b)] = \delta_{j,k} x_{i,l}(ab).$$

Note $ad(w)x_{i,j}(a) = wx_{i,j}(a)w^{-1} = x_{i+1,j+1}(a)$ identifying $i, j \in \mathbb{Z}/n$. Hence

$$InU^* = U^* \cap ad(w)U^* = \langle x_{i,j} | x_{1,j} = 0 \rangle$$

and $ad(w^{-1})InU^* = \langle x_{i,j} | x_{i,n} = 0 \rangle$, that is

$$InU^* = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & * & \dots & * \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots} & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \qquad ad(w^{-1})InU^* = \begin{pmatrix} 1 & * & \dots & * & 0 \\ 0 & 1 & \dots & * & 0 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} & \cdot \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

In Theorem 7.1, the InU^* action on U^* is given by the following ρ (see 1.11.4 in [De-Lu])

$$\rho(u)v = ad(\dot{w}_n^{-1})(u)vF(u^{-1}) \quad for\ u \in InU^*,\ v \in U^*.$$

Lemma 7.2. *There is an isomorphism*

$$U^*/\rho(InU^*) \cong \langle x_{ij}(a) | x_{i,j} = 0\ if\ j \neq n \rangle$$

$$= \{ \begin{pmatrix} 1 & 0 & \dots & 0 & d_1 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} & * \\ 0 & 0 & \dots & 1 & d_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in U^* \mid d_1, ..., d_{n-1} \in \bar{F}_p \}.$$

Proof. We consider the ρ-action when the case $u = x_{i,j}(a)$ and $v = x_{k,l}(b)$, namely,

$$\rho(u)v = ad(\dot{w}^{-1})(x_{ij}(a))x_{k,l}(b)F(x_{i,j}(a)^{-1})$$
$$= x_{i-1,j-1}(a)x_{k,l}(b)x_{i,j}(-a^q).$$

For roots $x_{i,j}$ and $x_{i',j'}$, we define an order $x_{i,j} < x_{i',j'}$ if $i < i'$ or $i = i'$, $j < j'$. Then any $v \in U^*$ is uniquely written by the product $\Pi x_{i,j}(b_{i,j})$ when we fix the above order in the product. For any $a \in U^*$, let x_{i_0,j_0} be the minimal root of v such that $x_{i_0,j_0}(b_{i_0,j_0}) \neq 0, j_0 < n$.

Take $i = i_0 + 1$, $j = j_0 + 1$ and $a = -b_{i_0,j_0}$. Then the equation

$$\rho(u)v = ad(\dot{w}^{-1})(x_{ij}(a))(\Pi x_{k,l}(b))F(x_{i,j}(a)^{-1})$$
$$= x_{i_0,j_0}(-b_{i_0,j_0})(\Pi x_{k,l}(b_{i,j}))x_{i_0+1,j_0+1}(-a^q)$$
$$= (\Pi_{(i_0,j_0)<(k,l)} x_{k,l}(b_{i,j}))x_{i_0+1,j_0+1}(-a^q)$$

implies that a nonzero minimal root of $\rho(u)v$ is larger than (i_0, j_0). Repeating this process, there exists $u \in InU^*$ such that $\rho(u)v = \Pi_{i=1}^{n-1} x_{i,n}(d_i)$.

But all nonzero elements in the right hand side group in this lemma are not in $Im(\rho(u))v$ for $u \neq 1$. Hence we have the lemma. $\square$

Recall that we can identify

$$Q' = \{x = \begin{pmatrix} x_1 & x_1^q & \dots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \dots & x_2^{q^{n-1}} \\ \dots & \dots & \dots & \dots \\ x_n & x_n^q & \dots & x_n^{q^{n-1}} \end{pmatrix} \in GL_n | |x|^{q-1} = det(x)^{q-1} = 1\}.$$

Theorem 7.3. *We can define the map* $f : Q' \to U^*/(\rho(InU^*)$ *by* $x \mapsto \dot{w}_n^{-1}x^{-1}Fx$*, in fact,*

$$f(x) = \begin{pmatrix} 1 & 0 & \dots & 0 & c_{n,1} \\ \dots & \dots & \dots & \dots & * \\ 0 & 0 & \dots & 1 & c_{n,n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where $c_{n,i} = c_{n,i}(x_1, ..., x_n)$ *is the Dickson element defined in §2. This map also induces the isomorphism*

$$G_n \backslash Q' \cong U^*/(\rho(InU^*)) \cong Spec(k[c_{n,1}, ..., c_{n,n-1}]) \qquad (so\ Q' \cong \tilde{X}(\dot{w}_n)).$$

Proof. Let us write

$$e_n \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix} = \begin{vmatrix} x_{j_1}^{q^{i_1}} & x_{j_1}^{q^{i_2}} & \dots & x_{j_1}^{q^{i_n}} \\ x_{j_2}^{q^{i_1}} & x_{j_2}^{q^{i_2}} & \dots & x_{j_2}^{q^{i_n}} \\ \dots & \dots & \dots & \dots \\ x_{j_n}^{q^{i_1}} & x_{j_n}^{q^{i_2}} & \dots & x_{j_n}^{q^{i_n}} \end{vmatrix}$$

so that $e_n \begin{pmatrix} 0 & 1 & \dots & n-1 \\ 1 & 2 & \dots & n \end{pmatrix} = e(x) = |x|$. Then the (j, i) cofactor of the matrix x is expressed as

$$D_{j,i} = (-1)^{i+j} e_{n-1} \begin{pmatrix} 0 & 1 & \dots & \hat{i-1} & \dots & n-1 \\ 1 & 2 & \dots & \hat{j} & \dots & n \end{pmatrix}.$$

By Clamer's theorem, we know

$$x^{-1} = |x|^{-1}(D_{j,i})^t = |x|^{-1}(D_{i,j}).$$

Let us write $(B_{i,j}) = |x|x^{-1}F(x)$. Then we can compute

$$B_{s,t} = \sum D_{s,k}x(k,t)^q = \sum D_{s,k}x_k^{q^t} \qquad (where\ x(k,l) = (k,l){-}entry\ of\ x)$$

$$= \begin{vmatrix} x_1 & \dots & \overset{s}{x_1^{q^t}} & \dots & x_1^{q^{n-1}} \\ x_2 & \dots & x_2^{q^t} & \dots & x_2^{q^{n-1}} \\ \dots & \dots & \dots & \dots & \dots \\ x_n & \dots & x_n^{q^t} & \dots & x_n^{q^{n-1}} \end{vmatrix}.$$

This element is nonzero only if $t = s-1$ or $t = n$. If $t = s-1$, then the above element is $|x|$. If $t = n$, then the above element is indeed,

$(-1)^{n-s}|x|c_{n,s-1}$ by the definition of the Dickson elements as stated in §2. Thus we have

$$x^{-1}F(x) = |x|^{-1}(B_{st}) = \begin{pmatrix} 0 & 0 & \dots & 0 & c_{n,0} \\ 1 & 0 & \dots & 0 & c_{n,1} \\ \multicolumn{4}{c}{\dots\dots\dots\dots} & * \\ 0 & 0 & \dots & 1 & c_{n,n-1} \end{pmatrix}.$$

Here $c_{n,0} = 1$ and acting $\dot{w}_n^{-1}$, we have the desired result for $f(x)$.

We will show that f is an isomorphism.
We note that f is decomposed into

$$\begin{tikzcd} G_n\backslash GL_n \arrow[r, "\bar{L}"] & GL_n \arrow[r, "\dot{w}_n^{-1}"] & G_n\backslash GL_n \\ G_n\backslash Q' \arrow[u, "incl."] \arrow[rr, "f"'] & & U^*/\rho(InU^*) \arrow[u, "incl."'] \end{tikzcd}$$

where $L(x) = x^{-1}F(x)$.

Since the Lang map is separable, so is f. We see that f is injective from the diagram. To show that f is an isomorphism, it is enough to see that $f : Q' \to U^*/\rho(InU^*)$ is surjective.
When we consider Q' as a subvariety of $\mathbb{A}^n$, the above f is identified with a map $g|Q'$, where $g : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is defined by $g(x) = (c_{n,1}(x), ..., c_{n,n-1}(x))$.
Then the surjectivity follows from the following lemma:

$\square$

Lemma 7.4. *Let $(f_1, ..., f_n)$ be a homogeneous regular sequence of $k[x_1, ..., x_n]$. Then the associated map $f : \mathbb{A}^n \to \mathbb{A}^n$ is surjective. It means that*
$f' : V(f_1 - a) \to \mathbb{A}^{n-1}$ is surjective for $a \in k$ where $f' = pr(f|V(f_1 - a))$ where $pr : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is the projection $pr(x_1, ..., x_n) = (x_1, ..., x_{n-1})$

Proof. We consider the inclusion $i : \mathbb{A}^n \subset \mathbb{P}^n$ defined by $i(x_1, ..., x_n) = [x_1, ..., x_n, 1]$, and denote the coordinate of $\mathbb{P}^n$by$[u_1, ...u_n, z] = [u, z]$. We denote by$\tilde{f} : \mathbb{P}^n \to \mathbb{P}^n$the rational map extended from f and denote by d_i the degree of f_i.

For $\alpha \in \mathbb{A}^n$,we see that $\tilde{f}^{-1}(\alpha)$is given by

$$V_+(f_1(u) - \alpha_1 z^{d_1}, ..., f_n(u) - \alpha_n z^{d_n}), when \alpha = (\alpha_1, ..., \alpha_n).$$

Then $\tilde{f}^{-1}(\alpha) \neq \phi$ by the Bezout theorem. Since $(f_1, ..., f_n)$ is a homogeneous regular sequence, we see that $V(f_1, ..., f_n) = \{0\}$. It implies that

$$\tilde{f}^{-1}(\alpha) \cap V_+(z) = \{[u_1, ..., u_n, 1] | f_1(u) = \cdots = f_n(u) = 0\} = \phi.$$

We have $f^{-1}(\alpha) = \tilde{f}^{-1}(\alpha) \neq \phi$.Hence f is surjective.

$\square$

Hence we know

$$\tilde{X}(\dot{w}) \cong \{(x_1, ..., x_n) \in \mathbb{A}^n | e(x_1, ..., x_n)^{q-1} = |x|^{q-1} = 1\}$$

Theorem 7.5. *There is an isomorphism of varieties*

$$X(1) \cong \tilde{X}(\dot{w}_n) \times_{T(\dot{w}_n)^F} \mathbb{G}_m.$$

Corollary 7.6. *We have isomorphisms*

$$G_n \backslash X(1) \cong G_n \backslash (\tilde{X}(\dot{w}_n) \times_{T(\dot{w})^F} \mathbb{G}_m \cong \mathbb{A}^{n-1} \times \mathbb{G}_m.$$

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